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Characteristic Value Problems Posed by Differential Equations Arising in Hydrodynamics and Hydromagnetics

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SUMMARY

A study is made of the general eigenvalue problem posed by a differential equation whose solutions must satisfy certain specified boundary conditions. It is shown that an adjoint eigenvalue problem can be defined and that, in general, it can be represented by a differential equation of the same order as that of the original system together with boundary conditions as numerous as those originally specified. This adjoint system generally differs from the original system but has the same eigenvalues and, moreover, each of its eigensolutions is orthogonal to every eigensolution of the original set except the one(s) belonging to the same eigenvalue. It is shown that the representation of this adjoint system is not, in general, unique, and, if it can be chosen to be identical with the original system, the problem is self-adjoint. A variational method is given for determining the eigenvalues.

To illustrate the application of the method, three problems are considered which are not self-adjoint: the stability of Couette flow, the onset of convection in a rotating sphere heated within, and self-maintained dynamo action in a conducting fluid of infinite extent.

I. INTRODUCTION

Within the past decade, Chandrasekhar has solved a large number of important stability problems in hydrodynamics and hydromagnetics. These have involved the calculation of characteristic values associated with differential equations that have sometimes been of a very high order. For many of these, he was able to express a characteristic value λ_i as the ratio $R(\phi_i)$ of two positive definite integrals in the corresponding characteristic function ϕ_i and he was thence able to show that, if ϕ_i is subjected to any small variation $\delta\phi_i$, the first order change δR_i in R_i is zero, and conversely. This forms the basis of a powerful and rapid method of determining the characteristic values: for each ϕ_i, assume a physically reasonable trial function, $\Phi_i(\alpha_1, \alpha_2 \ldots)$, which satisfies all the boundary conditions and all the continuity requirements within the region of integration and which, in addition, depends on a number of adjustable parameters $\alpha_1, \alpha_2, \ldots$ Evaluate $R(\Phi_i)$ and minimize the resulting expression with respect to $\alpha_1, \alpha_2, \ldots$ Then this value of $R(\Phi_i)$ is likely to be a good approximation to λ_i, since a first order error in the choice of Φ_i, gives rise to only a second order error in λ_i.

However, it is not always possible to express λ_i as a ratio of two positive definite integrals. Such characteristic value problems are not self-adjoint (in a sense to be defined presently), and are special objects of study in this paper. In § II, it is shown that they admit a variational principle and the method of solution based on this principle is compared with methods devised by Chandrasekhar. In §§ III, IV, and V illustrative examples are given.

II. Method of Solution

Let $\mathscr{L}$ be a linear differential operator defined within a volume $\mathscr{V}$ of the independent variables. Denote by $\mathscr{B}$ conditions on the boundary of $\mathscr{V}$ which define a characteristic value problem from $\mathscr{L}$, i.e.

$$\mathscr{L}\phi_i = \lambda_i \phi_i. \tag{1}$$

Here λ_i and ϕ_i denote a particular characteristic value and a characteristic function belonging to it. We will suppose that the set of functions $\{\phi_i\}$ spans $\mathscr{V}$. Define from $(\mathscr{L}, \mathscr{B})$ an *adjoint* operator and associated boundary conditions $(\mathscr{L}^*, \mathscr{B}^*)$ by the condition that, if f is any function satisfying $\mathscr{B}$ and f^* is any function satisfying $\mathscr{B}^*$, then

$$\int_{\mathscr{V}} f \mathscr{L}^* f^* = \int_{\mathscr{V}} f^* \mathscr{L} f. \tag{2}$$

The operator $\mathscr{L}^*$ defined in this way may be an abstract operator which cannot be represented simply in a differential form. Whether or not a differential form is possible depends on the suitability or otherwise of the conditions $\mathscr{B}^*$. If one repeatedly integrates the right-hand side of expression (2) by parts, and *chooses* each of the conditions $\mathscr{B}^*$ in turn in such a way that the integrated parts vanish either in virtue of $\mathscr{B}$ or in virtue of $\mathscr{B}^*$, then one must discover for $\mathscr{L}^*$ a *differential* form which is of the same order as $\mathscr{L}$. If, however, $\mathscr{B}^*$ is chosen in some way for which $\mathscr{L}^*$ is not a differential operator, it is nevertheless possible to attach a meaning to $\mathscr{L}^*$ (and, as we shall see later, to find a matrix representation for it), and the analysis we are about to give remains valid, although it is not so directly comprehensible. Naturally the representation $(\mathscr{L}^*, \mathscr{B}^*)$ is not unique and there may even be more than one representation in which $\mathscr{L}^*$ can assume a differential form. If, on choosing $\mathscr{B}^*$ to be $\mathscr{B}$, $\mathscr{L}^*$ is found to be $\mathscr{L}$, the characteristic value problem $(\mathscr{L}, \mathscr{B})$ is said to be *self-adjoint*. If not, the characteristic value problem adjoint to (1) is

$$\mathscr{L}^*\phi_i^* = \lambda_i^*\phi_i^*, \tag{1*}$$

where ϕ_i^* denotes a characteristic function satisfying (1*) and $\mathscr{B}^*$, and λ_i^* is the characteristic values to which it belongs. We will suppose that the set $\{\phi_i^*\}$ spans $\mathscr{V}$.

From equations (1), (1*) and (2), we have

$$\lambda_i \int_{\mathscr{V}} \phi_i\phi_j^* = \int_{\mathscr{V}} \phi_j^*\mathscr{L}\phi_i = \int_{\mathscr{V}} \phi_i\mathscr{L}^*\phi_j^* = \lambda_j^* \int_{\mathscr{V}} \phi_i\phi_j^*, \tag{3}$$

i.e.

$$(\lambda_i - \lambda_j^*) \int_{\mathscr{V}} \phi_i\phi_j^* = 0. \tag{4}$$

Hence, if λ_i and λ_j^* are unequal, we have

$$\int_{\mathscr{V}} \phi_i\phi_j^* = 0; \qquad \int_{\mathscr{V}} \phi_j^*\mathscr{L}\phi_i = 0. \tag{5}$$

Moreover, since $\{\phi_i\}$ spans $\mathscr{V}$, the equation

$$\int_{\mathscr{V}} \phi_i\phi_i^* = 0 \tag{6}$$

would, in conjunction with equation (5), imply that ϕ_i^* is identically zero, which is not so, by definition. Hence equation (4) implies that the sets of characteristic values $\{\lambda_i\}$ and $\{\lambda_i^*\}$ are identical. Moreover, the sets $\{\phi_i\}$ and $\{\phi_i^*\}$ are dual (or bi-orthonormal); that is, each function of either set is orthogonal to every member of the other set except the one(s) belonging to the same characteristic value. From equation (3) it also follows that

$$\lambda_i = \frac{\int_{\mathscr{V}} \phi_i^* \mathscr{L} \phi_i}{\int_{\mathscr{V}} \phi_i^* \phi_i}. \tag{7}$$

Let $\delta\phi_i$ and $\delta\phi_i^*$ be any two small (independent) variations which are applied to ϕ_i and ϕ_i^* and which satisfy the boundary conditions $\mathscr{B}$ and $\mathscr{B}^*$, respectively. The corresponding change $\delta\lambda_i$ in λ_i is

$$\delta\lambda_i = \frac{1}{\int_{\mathscr{V}} \phi_i^* \phi_i} \left[\left\{ \int_{\mathscr{V}} [\delta\phi_i^* \mathscr{L} \phi_i + \delta\phi_i \mathscr{L}^* \phi_i^*] \right\} - \frac{[\int_{\mathscr{V}} \phi_i^* \mathscr{L} \phi_i]}{[\int_{\mathscr{V}} \phi_i^* \phi_i]} \left\{ \int_{\mathscr{V}} [\phi_i^* \delta\phi_i + \phi_i \delta\phi_i^*] \right\} \right], \tag{8}$$

or, using equation (7),

$$\delta\lambda_i = \left[\int_{\mathscr{V}} \delta\phi_i^* [\mathscr{L}\phi_i - \lambda_i \phi_i] + \int_{\mathscr{V}} \delta\phi_i [\mathscr{L}^* \phi_i^* - \lambda_i \phi_i^*] \right] \Big/ \int_{\mathscr{V}} \phi_i^* \phi_i. \tag{9}$$

Hence, if equations (1) and (1*) are obeyed, $\delta\lambda_i$ is zero to first order for all small, arbitrary variations $\delta\phi_i$ and $\delta\phi_i^*$ satisfying $\mathscr{B}$ and $\mathscr{B}^*$, respectively. Evidently, the converse is also true. This forms the basis of a variational method which is analogous to the method described briefly in § 1 and reduces to it if the problem is self-adjoint. Provided the trial functions $\Phi_i(\alpha_1, \alpha_2, \ldots)$ and $\Phi_i^*(\alpha_1^*, \alpha_2^*, \ldots)$ are appropriately chosen and, of course, satisfy the boundary conditions $\mathscr{B}$ and $\mathscr{B}^*$, respectively, the error made in determining λ_i from equation (7) is small, being of second order in the error of Φ_i and Φ_i^*. The error can be reduced by determining the stationary value of λ_i under variations of $\alpha_1, \alpha_2, \ldots$ and $\alpha_1^*, \alpha_2^*, \ldots$ This stationary value will not, of course, be a minimum in general.

It remains to extend our discussion to the case in which the adjoint problem does not necessarily have a differential representation. We can find a matrix representation by taking two orthogonal bases $\{f_\alpha\}$ and $\{f_\alpha^*\}$ of $\mathscr{V}$ which satisfy $\mathscr{B}$ and $\mathscr{B}^*$, respectively. We associate with ϕ an infinite column vector $\{a_\beta\}$ by means of the expansion

$$\phi = \sum_{\beta=1}^{\infty} a_\beta f_\beta, \tag{10}$$

and we represent the operator $\mathscr{L}$ and the identity operator $\mathscr{I}$ by the infinite matrices $L_{\alpha\beta}$ and $I_{\alpha\beta}$ defined by

$$L_{\alpha\beta} = \int_{\mathscr{V}} f_\alpha^* \mathscr{L} f_\beta, \qquad I_{\alpha\beta} = \int_{\mathscr{V}} f_\alpha^* f_\beta. \tag{11}$$

Since $\{f_\alpha\}$ and $\{f_\alpha^*\}$ are orthogonal bases, we find that equation (1) reduces, in virtue of equations (10) and (11), to the matrix equation

$$\sum_{\beta=1}^{\infty} L_{\alpha\beta}\, a_\beta = \lambda \sum_{\beta=1}^{\infty} I_{\alpha\beta}\, a_\beta, \qquad \alpha = 1, 2, \ldots \tag{12}$$

The existence of solutions to these equations implies the existence of solutions, having the *same* characteristic values, to the equations

$$\sum_{\beta=1}^{\infty} L_{\beta\alpha}\, a_\beta^* = \lambda \sum_{\beta=1}^{\infty} I_{\beta\alpha}\, a_\beta^*, \qquad \alpha = 1, 2, \ldots \tag{12*}$$

The characteristic values themselves are the roots of the equation

$$||L_{\alpha\beta} - \lambda I_{\alpha\beta}|| = 0, \tag{13}$$

where the vertical double lines at each side denote the infinite determinant whose element in row α and column β is given by the expression between them. Equations (12*) are the matrix representation of the adjoint system, the characteristic functions of which are

$$\phi^* = \sum_{\beta=1}^{\infty} a_\beta^* f_\beta^*. \tag{10*}$$

One of the methods used by Chandrasekhar for problems which are not self-adjoint is that of solving equation (13) approximately. To find any particular characteristic value, he picks the base function f_1 (say) which is physically the most plausible and adds a few (say, $n - 1$) of the adjacent functions $f_2, f_3, \ldots f_n$ (say) to form a finite approximation to ϕ:

$$\phi = \sum_{\beta=1}^{n} a_\beta f_\beta. \tag{14}$$

Equation (13) now reduces to an $n \times n$ determinant and, after solving this for the appropriate characteristic value $\lambda(n)$, he examines the behaviour of $\lambda(n)$ as a function of n. If $\lambda(n)$ appears to converge to a limit $\lambda(\infty)$, say, he presumes that this is a good approximation to the required characteristic value. Very often, the convergence is rapid and leads speedily to a good estimate of λ. That this is so is not surprising. We can regard equation (14) and the corresponding approximation

$$\phi^* = \sum_{\beta=1}^{n} a_\beta^* f_\beta^* \tag{14*}$$

as trial functions for the variational principle based on equation (7). The best values of $\{a_\alpha\}$ and $\{a_\alpha^*\}$ are those for which

$$\partial\lambda/\partial a_\alpha = \partial\lambda/\partial a_\alpha^* = 0, \qquad \alpha = 1, 2, \ldots n, \tag{15}$$

that is, those for which

$$\sum_{\beta=1}^{n} L_{\alpha\beta}\, a_\beta = \lambda \sum_{\beta=1}^{n} I_{\alpha\beta}\, a_\beta, \qquad \alpha = 1, 2, \ldots n, \tag{16}$$

$$\sum_{\beta=1}^{n} L_{\beta\alpha}\, a_\beta^* = \lambda \sum_{\beta=1}^{n} I_{\beta\alpha}\, a_\beta^*, \qquad \alpha = 1, 2, \ldots n. \tag{16*}$$

Elimination of a_β or a_β^* leads to the determinant equation of order n which determines $\lambda(n)$.

It is appropriate here to describe another method frequently used by Chandrasekhar. Its success depends on the introduction of auxiliary variables with the aid of which λ may be expressed as the ratio of two positive definite integrals despite the fact that the problem is not self-adjoint. In essence, it is based on the possibility of factorizing $\mathscr{L}(=\mathscr{N}\mathscr{M})$ into two non-commuting operators $\mathscr{M}, \mathscr{N}$ each of which is self-adjoint even though their product is not. If such a division is possible, the boundary conditions $\mathscr{B}$ can be separated into two groups $\mathscr{C}$ and $\mathscr{D}$ in such a way that the differential equation

$$\mathscr{N}\psi = \phi \tag{17}$$

together with $\mathscr{D}$ determine ψ from ϕ uniquely. The characteristic value problem can now be expressed as the solution of the differential equation

$$\mathscr{M}\phi = \lambda\psi \tag{18}$$

subject to the boundary conditions $\mathscr{C}$. The statement we have made that $\mathscr{M},\mathscr{N}$ are self-adjoint means in this context that, if f_1 and f_2 are any two functions satisfying $\mathscr{C}$, and g_1 and g_2 are any two functions satisfying $\mathscr{D}$, then

$$\int_{\mathscr{V}} f_1\mathscr{M}f_2 = \int_{\mathscr{V}} f_2\mathscr{M}f_1; \qquad \int_{\mathscr{V}} g_1\mathscr{N}g_2 = \int_{\mathscr{V}} g_2\mathscr{N}g_1. \tag{19}$$

According to equation (18),

$$\lambda = \int_{\mathscr{V}} \phi\mathscr{M}\phi \bigg/ \int_{\mathscr{V}} \psi\mathscr{N}\psi, \tag{20}$$

and so, on a small variation $\delta\phi$, the corresponding change in λ is

$$\delta\lambda = 2\left[\int_{\mathscr{V}} \delta\phi\mathscr{M}\phi - \lambda\int_{\mathscr{V}} \psi\mathscr{N}\delta\psi\right] \bigg/ \int \psi\mathscr{N}\psi, \tag{21}$$

and, according to the definition of ψ, the corresponding change in $\delta\psi$ satisfies (cf. equation (17))

$$\mathscr{N}\delta\psi = \delta\phi. \tag{22}$$

Hence

$$\delta\lambda = 2\int_{\mathscr{V}} \delta\phi(\mathscr{M}\phi - \lambda\psi) \bigg/ \int_{\mathscr{V}} \psi\mathscr{N}\psi, \tag{23}$$

from which the variational principle follows. It is a principle of the conventional type in which λ is expressed as the ratio of two positive definite forms. However, it is clear that, according to equation (2),

$$\mathscr{L}^* = (\mathscr{N}\mathscr{M})^* = \mathscr{M}^*\mathscr{N}^* = \mathscr{M}\mathscr{N}, \tag{24}$$

and this is not equal to $\mathscr{L}(=\mathscr{N}\mathscr{M})$ since $\mathscr{M},\mathscr{N}$ do not commute. Thus, the characteristic value problem is not self-adjoint but, provided the auxiliary variable ψ is introduced, the variational principle nevertheless involves the ratio of two positive definite forms.

In the following sections, we consider three particular problems which are not self-adjoint; namely, the Taylor problem of the stability of viscous flow between rotating cylinders, and the onset of convection in a rotating fluid sphere heated within, and a modified form of the dynamo problem.

III. Stability of Couette Flow

From the mathematical viewpoint, this example is particularly suited for demonstrating the non-uniqueness of the adjoint system itself. For simplicity, we will limit the discussion to the case in which the separation between the rotating cylinders is small compared to their radii. The method can, however, be extended fairly easily to the more general case. We consider, then, the characteristic value problem posed by the differential equation

$$(D^2 - a^2)^3 v = -a^2 \operatorname{Ta}(1 + \alpha z) v, \tag{25}$$

whose solutions are subjected to the boundary conditions

$$v = (D^2 - a^2)v = D(D^2 - a^2)v = 0, \qquad \text{at} \qquad z = 0, 1, \tag{26}$$

where Ta, the Taylor number, is the characteristic value parameter, $D = d/dz$, and $a, \alpha (0 \geqslant \alpha \geqslant -3.0)$ are real constants [2, 3]. Chandrasekhar has shown that the best numerical results can be obtained by splitting equation (25) into two separate equations; a fourth order one (in an auxiliary variable ϕ) which is solved exactly, and a second order one which is solved by other means. We write, therefore,

$$\mathscr{L}\colon \begin{cases} (D^2 - a^2)^2 \phi = (1 + \alpha z) v, & (27) \\ (D^2 - a^2) v \;\; = -a^2 \operatorname{Ta} \phi, & (28) \end{cases}$$

where

$$\mathscr{B}\colon\ \phi = D\phi = v = 0, \qquad \text{at} \qquad z = 0, 1. \tag{29}$$

The adjoint problem can be found quite systematically by the method explained in § II using the orthogonal base of functions

$$f_n = f_n^* = \sin n\pi z, \qquad n = 1, 2, \ldots \tag{30}$$

It is then easily shown that, in this matrix representation, the problem reduces to an infinite set of linear equations equivalent to a set already derived by Chandrasekhar [3, eq. (30)]. However, much the simplest way of defining an adjoint problem is by direct integration by parts in the manner described in § II (eq. 2 *et seq.*). We will show, in fact, that the adjoint problem may be expressed as

$$\mathscr{L}^*: \begin{cases} (D^2 - a^2)^2 \phi^* = v^*, & (27^*) \\ (D^2 - a^2)v^* = -a^2 \,\mathrm{Ta}^* \,(1 + \alpha z)\phi^*, & (28^*) \end{cases}$$

where

$$\mathscr{B}^* = \mathscr{B}:\ \phi^* = D\phi^* = v^* = 0, \qquad \text{at} \qquad z = 0, 1. \tag{29*}$$

To prove this, consider the integral

$$I = \int_0^1 [(Dv)(Dv^*) + a^2 vv^*]\, dz \tag{31}$$

evaluated in two different ways. Integrating by parts using equations (27*), (28), (29*), we find

$$I = -\int_0^1 v^*(D^2 - a^2)v\, dz = a^2\,\mathrm{Ta}\int_0^1 \phi(D^2 - a^2)^2 \phi^*\, dz. \tag{32}$$

Integrating by parts twice, using equations (29), we obtain

$$I = a^2\,\mathrm{Ta}\int_0^1 [(D^2 - a^2)\phi][(D^2 - a^2)\phi^*]\, dz. \tag{33}$$

Alternatively, integrating I by parts using equations (27), (28*), and (29) gives

$$I = -\int_0^1 v(D^2 - a^2)v^*\, dz = a^2\,\mathrm{Ta}^*\int_0^1 \phi^*(D^2 - a^2)^2 \phi\, dz. \tag{32*}$$

Integrating by parts twice more, using equations (29*), we obtain

$$I = a^2\,\mathrm{Ta}^*\int_0^1 [(D^2 - a^2)\phi][(D^2 - a^2)\phi^*]\, dz. \tag{33*}$$

Comparing equations (33) and (33*), we see that I vanishes unless Ta and Ta* are equal, and also that

$$\mathrm{Ta} = \frac{\int_0^1 [(Dv)(Dv^*) + a^2 vv^*]\, dz}{a^2 \int_0^1 [(D^2 - a^2)\phi][(D^2 - a^2)\phi^*]\, dz}. \tag{34}$$

That this forms the basis of a variational principle follows from an argument too close to that given in § II to be worth repeating here.

It should be noticed that the set (27*) to (29*) is not the only representation of the adjoint problem in terms of a differential operator. The set given corresponds to the *same* boundary conditions (26) as those of the original problem but a *different* differential equation, namely

$$(D^2 - a^2)^2 \frac{1}{(1 + \alpha z)} (D^2 - a^2) v^* = - a^2 \,\mathrm{Ta}\, v^*. \tag{35}$$

However, it is clear from equations (27*) to (29*) that we can also represent the adjoint problem by the *same* differential equation (25) but with *different* boundary conditions, namely

$$v^* = Dv^* = (D^2 - a^2)^2 v^* = 0, \qquad \text{at} \qquad z = 0, 1, \tag{36}$$

which, are, in a sense conditions complementary to (26). It is obviously also possible to represent the adjoint problem in yet another way in which neither differential equation nor boundary conditions are the same, namely

$$(D^2 - a^2) \frac{1}{(1 + \alpha z)} (D^2 - a^2)^2 v^* = - a^2 \,\mathrm{Ta}\, v^*, \tag{37}$$

where

$$(D^2 - a^2) v^* = (D^2 - a^2)^2 v^* = D(D^2 - a^2)^2 v^* = 0, \qquad \text{at} \qquad z = 0, 1. \tag{38}$$

IV. Convection in a Rotating Sphere

From the mathematical viewpoint, this example is particularly suited for demonstrating that, with but one representation of the adjoint problem, it is possible to find several distinct variational principles differing in the choice of the auxiliary variables. In suitably chosen units, the stationary states of marginal stability of convection in a rotating sphere are governed by the equations

$$\mathscr{L}: \begin{cases} \nabla^2\Theta + \mathbf{u}\cdot\mathbf{r} = 0, & (39) \\ \mathrm{Ra}\,\Theta\mathbf{r} = \mathrm{curl}^2\,\mathbf{u} + \mathrm{Ta}^{1/2}\,\mathbf{1}_z \times \mathbf{u} + \mathrm{grad}\,\tilde{\omega}, & (40) \\ \mathrm{div}\,\mathbf{u} = 0, & (41) \end{cases}$$

and the boundary conditions

$$\mathscr{B}: \begin{cases} \Theta = 0, \quad \text{at} \quad r = 1, & (42) \\ \mathbf{u} = 0, \quad \text{at} \quad r = 1. & (43) \end{cases}$$

Here Θ denotes the temperature fluctuation; $\tilde{\omega}$, the pressure fluctuation; $\mathbf{u}$, the fluid velocity; $\mathbf{r}$, the coordinate vector from the centre of the sphere; and $\mathbf{1}_z$, a unit vector along the axis of rotation. The conditions (43) are appropriate to a rigid boundary at the surface of the sphere; the case of a free boundary poses no extra problems but is slightly more cumbersome algebraically. Ra and Ta denote the Rayleigh number and Taylor number, respectively. Full details of the derivation of these equations are given by Chandrasekhar [4]. If Ta is non-zero, the characteristic value problem is not self-adjoint. We will show that a representation of the adjoint system is

$$\mathscr{L}^*: \begin{cases} \nabla^2\Theta^* + \mathbf{u}^*\cdot\mathbf{r} = 0, & (39^*) \\ \mathrm{Ra}^*\,\Theta^*\mathbf{r} = \mathrm{curl}^2\,\mathbf{u}^* - \mathrm{Ta}^{1/2}\,\mathbf{1}_z \times \mathbf{u}^* + \mathrm{grad}\,\tilde{\omega}^*, & (40^*) \\ \mathrm{div}\;\mathbf{u}^* = 0, & (41^*) \end{cases}$$

together with the same boundary conditions

$$\mathscr{B}^* = \mathscr{B}: \begin{cases} \Theta^* = 0, \quad \text{at} \quad r = 1, & (42^*) \\ \mathbf{u}^* = 0, \quad \text{at} \quad r = 1. & (43^*) \end{cases}$$

That Ra and Ra* are equal is physically obvious, since the equations (39*) to (43*) correspond to the original problem with an opposite sense of rotation, i.e., taking spherical polar coordinates (r, θ, ϕ) with $\mathbf{1}_z$ as axis, if

$$\Theta = \Theta(r, \theta, \phi), \qquad \mathbf{u} = [u_r(r, \theta, \phi), u_\theta(r, \theta, \phi), u_\phi(r, \theta, \phi)], \tag{44}$$

satisfies equations (39) to (43), then

$$\Theta^* = \Theta(r, \theta, -\phi), \mathbf{u}^* = [u_r(r, \theta, -\phi), u_\theta(r, \theta, -\phi), -u_\phi(r, \theta, -\phi)], \tag{44*}$$

satisfies equations (39*) to (43*), provided we set Ra = Ra*. To prove that equations (39*) to (43*) do, indeed, define the adjoint problem, consider the integral

$$I = \int (\operatorname{grad} \Theta) \cdot (\operatorname{grad} \Theta^*) \, d\tau \tag{45}$$

evaluated in two different ways. By the divergence theorem and equations (39) and (42*), we have

$$I = -\int \Theta^* \nabla^2 \Theta \, d\tau = \int \mathbf{u} \cdot \Theta^* \mathbf{r} \, d\tau. \tag{46}$$

Using now equations (40*), (41*), (43) in conjunction with the divergence theorem, we find

$$I = \frac{1}{\mathrm{Ra}} \left\{ \int (\operatorname{curl} \mathbf{u}) \cdot (\operatorname{curl} \mathbf{u}^*) \, d\tau + \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot (\mathbf{u} \times \mathbf{u}^*) \, d\tau \right\}. \tag{47}$$

Alternatively, by the divergence theorem and equations (39*) and (42), we have

$$I = -\int \Theta \nabla^2 \Theta^* \, d\tau = \int \mathbf{u}^* \cdot \Theta \mathbf{r} \, d\tau. \tag{46*}$$

Using now equations (40), (41), (43*) in conjunction with the divergence theorem, we find

$$I = \frac{1}{\mathrm{Ra}^*} \left\{ \int (\operatorname{curl} \mathbf{u}) \cdot (\operatorname{curl} \mathbf{u}^*) \, d\tau - \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot (\mathbf{u}^* \times \mathbf{u}) \, d\tau \right\}. \tag{47*}$$

Comparing equations (47) and (47*), we see that I vanishes unless Ra and Ra* are equal, and also that

$$\mathrm{Ra} = \frac{\int (\operatorname{curl} \mathbf{u}) \cdot (\operatorname{curl} \mathbf{u}^*) \, d\tau + \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot (\mathbf{u} \times \mathbf{u}^*) \, d\tau}{\int (\operatorname{grad} \Theta) \cdot (\operatorname{grad} \Theta^*) \, d\tau} \tag{48}$$

If Ta is zero, the problem is self-adjoint, and equation (48) leads to the variational principle discovered by Chandrasekhar [**1**] for that case. The proof that the expression (48) forms the basis of a variational method in the more general case contains a point of divergence from the proof of § II, and will, therefore, be considered below. The attitude we take is that Θ and Θ^* are auxiliary variables determined from $\mathbf{u}$ and $\mathbf{u}^*$ by the subsidiary equations (39) and (39*) and the conditions (42) and (42*).

Thus, when we make small independent solenoidal variations $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$ to $\mathbf{u}$ and $\mathbf{u}^*$, we will not suppose that $\delta\Theta$ and $\delta\Theta^*$ are arbitrary, but we will consider that they are determined from $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$ by equations (39), (39*), (42), (42*).

According to equation (48), the first order change $\delta\mathrm{Ra}$ in Ra produced by the small variations $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$ is

$$\delta\,\mathrm{Ra} = \frac{1}{I}\left[\int\{[(\mathrm{curl}\,\mathbf{u})\cdot(\mathrm{curl}\,\delta\mathbf{u}^*) + \mathrm{Ta}^{1/2}\,\mathbf{1}_z\cdot(\mathbf{u}\times\delta\mathbf{u}^*) - \mathrm{Ra}\,(\mathrm{grad}\,\Theta)\cdot(\mathrm{grad}\,\delta\Theta^*)] + [(\mathrm{curl}\,\mathbf{u}^*)\cdot(\mathrm{curl}\,\delta\mathbf{u}) - \mathrm{Ta}^{1/2}\,\mathbf{1}_z\cdot(\mathbf{u}^*\times\delta\mathbf{u}) - \mathrm{Ra}\,(\mathrm{grad}\,\Theta^*)\cdot(\mathrm{grad}\,\delta\Theta)]\}\,d\tau\right]. \tag{49}$$

On applying the divergence theorem, using $\mathscr{B}$ and $\mathscr{B}^*$ and equations (39), (39*) to relate $\delta\Theta$ and $\delta\Theta^*$ to $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$, we find

$$\delta\,\mathrm{Ra} = \frac{1}{I}\left[\int\{\delta\mathbf{u}^*\cdot[\mathrm{curl}^2\,\mathbf{u} + \mathrm{Ta}^{1/2}\,(\mathbf{1}_z\times\mathbf{u}) - \mathrm{Ra}\,\Theta\mathbf{r}] + \delta\mathbf{u}\cdot[\mathrm{curl}^2\,\mathbf{u}^* - \mathrm{Ta}^{1/2}\,(\mathbf{1}_z\times\mathbf{u}^*) - \mathrm{Ra}\,\Theta^*\mathbf{r}]\}\,d\tau\right]. \tag{50}$$

Hence, if $\mathbf{u}$ and $\mathbf{u}^*$ satisfy equations (40) and (40*), then Ra is zero for all independent *solenoidal* variations $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$. Conversely, if $\delta\mathrm{Ra}$ vanishes for all independent *solenoidal* variations $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$ satisfying the boundary conditions (43) and (43*), then there exist single-valued functions $\tilde{\omega}$ and $\tilde{\omega}^*$ such that

$$\mathrm{curl}^2\,\mathbf{u} + \mathrm{Ta}^{1/2}\,\mathbf{1}_z\times\mathbf{u}^* - \mathrm{Ra}\,\Theta = -\,\mathrm{grad}\,\tilde{\omega}, \tag{40}$$

$$\mathrm{curl}^2\,\mathbf{u}^* - \mathrm{Ta}^{1/2}\,\mathbf{1}_z\times\mathbf{u}^* - \mathrm{Ra}\,\Theta^* = -\,\mathrm{grad}\,\tilde{\omega}^*. \tag{40*}$$

Note that we must use the facts that $\delta\mathbf{u}$ and $\delta\mathbf{u}^*$ are solenoidal. This reflects the fact that the characteristic vectors $\mathbf{u}(\mathbf{u}^*)$ satisfying $\mathscr{L}$ and $\mathscr{B}$ ($\mathscr{L}^*$ and $\mathscr{B}^*$) do not form a base by which *any* vector satisfying equation (43) (equation 43*) can be expanded, but only a base by which any such *solenoidal* vector can be expanded.

The formulation of a variational principle is not unique. Let us divide $\mathbf{u}$ into its toroidal and poloidal parts

$$\mathbf{u} = \mathbf{u}_T + \mathbf{u}_S = \mathrm{curl}\,T\mathbf{r} + \mathrm{curl}^2\,S\mathbf{r} \tag{51}$$

It may be shown, in a straightforward manner, that an arbitrary vector field $\mathbf{A}$ may be written in the form

$$\mathbf{A} = \mathrm{grad}\,U' + V'\mathbf{r} + \mathrm{curl}\,W'\mathbf{r}, \tag{52}$$

where

$$L^2U' = \left(1 + x_i \frac{\partial}{\partial x_i}\right) x_j A_j - r^2 \frac{\partial A_i}{\partial x_i}, \tag{53}$$

$$L^2V' = \frac{\partial}{\partial x_j}\left(1 + x_i \frac{\partial}{\partial x_i}\right) A_j - \nabla^2 x_i A_i,$$

$$L^2W' = \mathbf{r} \cdot \operatorname{curl} \mathbf{A},$$

and L^2 is a second order differential operator which commutes with ∇^2, and which can be expressed in any of the forms

$$L^2 = \left(x_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_i} x_j \frac{\partial}{\partial x_j} - r^2 \nabla^2\right) = r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \nabla^2\right) \tag{54}$$

$$= - \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

It therefore follows, writing $\mathbf{1}_z \times \mathbf{u}$ for $\mathbf{A}$, that

$$\mathbf{1}_z \times \mathbf{u} = \operatorname{grad} U + V\mathbf{r} + \operatorname{curl} W\mathbf{r}, \tag{55}$$

where

$$L^2U = r\left[\sin\theta \frac{\partial}{\partial\theta} + \cos\theta L^2\right] T - \frac{\partial}{\partial\phi}\left[\frac{\partial}{\partial r}(rS) + L^2S\right], \tag{56}$$

$$L^2V = Q^3T + \frac{\partial}{\partial\phi}\nabla^2 S,$$

$$L^2W = Q^3S - \frac{\partial T}{\partial\phi}$$

and where Q^3 is a third order differential operator:

$$Q^3 = \frac{1}{r\sin\theta}\left\{\left[\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial}{\partial\theta}\right) + \cot\theta\frac{\partial^2}{\partial\phi^2}\right]\left(r\frac{\partial}{\partial r} + 1\right) - \right. \tag{57}$$

$$\left. \left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}$$

$$= -\frac{1}{r}\left\{\left(\sin\theta\frac{\partial}{\partial\theta} + \cos\theta L^2\right)\left(r\frac{\partial}{\partial r} + 1\right) - L^2\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}.$$

It is clear, by integrating three times by parts, that

$$\int AQ^3B\,d\tau = -\int BQ^3A\,d\tau, \tag{58}$$

provided the integrated parts vanish (as they do for the functions A, B to which equation (58) is applied below).

Applying equations (51), (55) and (56) to equations (39) and (40), we find

$$\nabla^2\Theta = -L^2S, \tag{59}$$

$$\nabla^2\left[\nabla^2L^2S + \mathrm{Ta}^{1/2}\frac{\partial S}{\partial\phi}\right] + \mathrm{Ta}^{1/2}Q^3T = \mathrm{Ra}\,L^2\Theta, \tag{60}$$

$$\left[\nabla^2L^2T + \mathrm{Ta}^{1/2}\frac{\partial T}{\partial\phi}\right] - \mathrm{Ta}^{1/2}Q^3S = 0, \tag{61}$$

together with another equation which relates $\tilde{\omega}$, T, S and which is of no significance here. The boundary conditions on Θ, S, T are

$$\Theta = S = T = \partial S/\partial r = 0, \qquad \text{at} \qquad r = 1. \tag{62}$$

As before the adjoint problem can be represented by the same boundary conditions and by equations derived from (59) to (62) by reversing the sign of $\mathrm{Ta}^{1/2}$. This implies, again, that one solution to the adjoint system is

$$\Theta^*(r, \theta, \phi) = \Theta(r, \theta, -\phi); \qquad S^*(r, \theta, \phi) = S(r, \theta, -\phi); \tag{63}$$
$$T^*(r, \theta, \phi) = -T(r, \theta, -\phi).$$

Before proceeding further, notice that, from the equation adjoint to (59), the boundary conditions (62), and applications of the divergence theorem, that

$$\int S^*L^2\Theta\,d\tau = \int (\operatorname{grad}\Theta)\cdot(\operatorname{grad}\Theta^*)\,d\tau \tag{64}$$

By applications of the divergence theorem and use of equations (60) and (64), we find

$$\int (\nabla^2S^*)(L^2\nabla^2S)\,d\tau = \int S^*\nabla^4L^2S\,d\tau \tag{65}$$

$$= \mathrm{Ra}\int S^*L^2\Theta\,d\tau - \mathrm{Ta}^{1/2}\int S^*\frac{\partial}{\partial\phi}\nabla^2S\,d\tau - \mathrm{Ta}^{1/2}\int S^*Q^3T\,d\tau$$

$$= \mathrm{Ra}\int (\operatorname{grad}\Theta)\cdot(\operatorname{grad}\Theta^*)\,d\tau - \mathrm{Ta}^{1/2}\int S^*\frac{\partial}{\partial\phi}\nabla^2S\,d\tau - \mathrm{Ta}^{1/2}\int S^*Q^3T\,d\tau,$$

a result which may also be written

$$\int (\operatorname{curl} \mathbf{u}_S) \cdot (\operatorname{curl} \mathbf{u}_S^*)\, d\tau = \mathrm{Ra} \int (\operatorname{grad} \Theta) \cdot (\operatorname{grad} \Theta^*)\, d\tau - \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot (\mathbf{u} \times \mathbf{u}_S^*)\, d\tau. \tag{66}$$

Similarly, by the divergence theorem and equation (61*), we find

$$\int TL^2\nabla^2 T^*\, d\tau = \mathrm{Ta}^{1/2} \int T \frac{\partial T^*}{\partial \phi}\, d\tau - \mathrm{Ta}^{1/2} \int TQ^3 S^*\, d\tau, \tag{67}$$

a result which may also be written

$$\int (\operatorname{curl} \mathbf{u}_T) \cdot (\operatorname{curl} \mathbf{u}_T^*)\, d\tau = - \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot (\mathbf{u} \times \mathbf{u}_T^*)\, d\tau. \tag{68}$$

By *adding* equations (66) and (68), we recover the variational principle (48) which, as we have already remarked, is not self-adjoint and gives a stationary value for Ra for all independent variations δS, δS^*, δT, δT^* when S, T satisfies equations (60) and (61). By subtracting equations (45) and (46), we find

$$\mathrm{Ra} \int (\operatorname{grad} \Theta) \cdot (\operatorname{grad} \Theta^*)\, d\tau = \int [\operatorname{curl} (\mathbf{u}_S + \mathbf{u}_T)] \cdot [\operatorname{curl} (\mathbf{u}_S^* - \mathbf{u}_T^*)]\, d\tau + \mathrm{Ta}^{1/2} \int \mathbf{1}_z \cdot [(\mathbf{u}_S + \mathbf{u}_T) \times (\mathbf{u}_S^* - \mathbf{u}_T^*)\, d\tau. \tag{69}$$

It can be shown that this also forms the basis for a variational principle which is not self-adjoint in general. Expression (69) gives a stationary value for Ra for all independent variations δS and δS^* when equations (60) and (60*) are satisfied, *provided* T and T^* are regarded as auxiliary variables *defined* from S and S^* by equations (61) and (61*) (and the boundary conditions that T and T^* vanish at the surface of the sphere).

For solutions independent of ϕ, equation (63) shows that

$$\mathbf{u}_S + \mathbf{u}_T = \mathbf{u}_S^* - \mathbf{u}_T^* = \mathbf{u}, \qquad \Theta^* = \Theta, \tag{70}$$

and hence equation (69) gives rise to a variational principle which is self-adjoint (cf. *5, 6*):

$$\mathrm{Ra} = \frac{\int (\operatorname{curl} \mathbf{u})^2\, d\tau}{\int (\operatorname{grad} \Theta)^2\, d\tau}. \tag{71}$$

This has the familiar physical interpretation that, in a state of marginal stability, there is a balance between the rate at which viscosity degrades hydrodynamic energy into heat and the rate at which buoyancy forces do work. However, we reiterate that equation (69) can form the basis of a variational principle, only when we interpret equation (61) as a restraint imposed by the rotation. This was unnecessary for the principle (48).

Even in the case where solutions are sought which are not independent of ϕ, it is possible to derive a variational principle which is self-adjoint by the application of alternative constraints. Write

$$\left.\begin{aligned} \Theta &= \Theta_1 \cos m\phi + \Theta_2 \sin m\phi \\ S &= S_1 \cos m\phi + S_2 \sin m\phi \\ T &= T_1 \cos m\phi + T_2 \sin m\phi \end{aligned}\right\} \tag{72}$$

Then, by equations (59) to (61), we have

$$\nabla^2\Theta_1 = - L^2 S_1, \tag{73}$$

$$\nabla^2\Theta_2 = - L^2 S_2, \tag{74}$$

$$\nabla^2[\nabla^2 L^2 S_1 + m \,\mathrm{Ta}^{1/2} S_2] = \mathrm{Ra}\, L^2 \Theta_1 - \mathrm{Ta}^{1/2} Q^3 T_1, \tag{75}$$

$$\nabla^2[\nabla^2 L^2 S_2 - m \,\mathrm{Ta}^{1/2} S_1] = \mathrm{Ra}\, L^2 \Theta_2 - \mathrm{Ta}^{1/2} Q^3 T_2, \tag{76}$$

$$\nabla^2 L^2 T_1 + m \,\mathrm{Ta}^{1/2} T_2 = \mathrm{Ta}^{1/2} Q^3 S_1, \tag{77}$$

$$\nabla^2 L^2 T_2 - m \,\mathrm{Ta}^{1/2} T_1 = \mathrm{Ta}^{1/2} Q^3 S_2, \tag{78}$$

where ∇^2, L^2, Q^3 are now to be understood to have their old meanings with $\partial^2/\partial\phi^2$ replaced by $-m^2$. It follows from equations (73) to (78) that

$$\mathrm{Ra}\int [(\mathrm{grad}\,\Theta_1)^2 + (\mathrm{grad}\,\Theta_2)^2]\,d\tau = \int [S_1\nabla^4 L^2 S_1 + S_2 \nabla^4 L^2 S_2 - T_1 \nabla^2 L^2 T_1 - T_2 \nabla^2 L^2 T_2]\,d\tau \tag{79}$$

and this is identical with equation (71). It can easily be verified that this defines a variational principle which is self-adjoint. Expression (79) gives an extreme value of Ra for all independent variations $\delta\Theta_2$, δS_1 and δT_2 when equations (74), (75) and (78) are satisfied *provided* Θ_1, S_2 and T_1 are regarded as auxiliary variables *defined* from Θ_2, S_1 and T_2 by equations (73), (76) and (77) and the appropriate boundary conditions. (The same is true, *mutatis mutandis*, if Θ_2, S_1 and T_2 are regarded as subsidiary variables defined from Θ_1, S_2 and T_1.)

Of course, as it stands the determination of Θ_1, S_2 and T_1 from equations (73), (76) and (77) involves a prior knowledge of Ra. However, this is simply rectified by a change of variable which alters the auxiliary problem into

$$\nabla^2 \Theta_1 = - L^2 S_1 \tag{80}$$

$$\nabla^2(\nabla^2 L^2 S_2 - m S_1) = L^2 \Theta_2 - Q^3 T_2 \tag{81}$$

$$\nabla^2 L^2 T_1 + m T_2 = Q^3 S_1 \tag{82}$$

$$\Theta_1 = S_2 = T_1 = \partial S_2/\partial r = 0, \qquad \text{at} \qquad r = 1, \tag{83}$$

the characteristic value problem into

$$\nabla^2 \Theta_2 = - \mathrm{Ra}\, L^2 S_2 \tag{84}$$

$$\nabla^2(\nabla^2 L^2 S_1 + m\, \mathrm{Ta}\, S_2) = \mathrm{Ra}\, L^2 \Theta_1 - \mathrm{Ta}\, Q^3 T_1 \tag{85}$$

$$\nabla^2 L^2 T_2 - m\, \mathrm{Ta}\, T_1 = \mathrm{Ta}\, Q^3 S_2 \tag{86}$$

$$\Theta_2 = S_1 = T_2 = \partial S_1/\partial r = 0, \qquad \text{at} \qquad r = 1, \tag{87}$$

and the basis of the variational principle into

$$\mathrm{Ra} \int (\mathrm{grad}\, \Theta_1)^2\, d\tau + \frac{\mathrm{Ta}}{\mathrm{Ra}} \int (\mathrm{grad}\, \Theta_2)^2\, d\tau \tag{88}$$

$$= \int (S_1 \nabla^4 L^2 S_1 - T_2 \nabla^2 L^2 T_2)\, d\tau + \mathrm{Ta} \int (S_2 \nabla^4 L^2 S_2 - T_1 \nabla^2 L^2 T_1)\, d\tau.$$

V. The Dynamo Problem for a Conducting Fluid of Infinite Extent

The characteristic value problem governing the existence of a steady self-maintained dynamo action in a fluid of infinite extent is

$$\mathscr{L}: \begin{cases} \mathrm{curl}\, \mathbf{B} = \mathrm{Rm}\,(-\,\mathrm{grad}\, \Phi + \mathbf{u} \times \mathbf{B}), & (89) \\ \mathrm{div}\, \mathbf{B} = 0, & (90) \end{cases}$$

and, at great distances (R),

$$\mathscr{B}: \quad \mathbf{B} = O(R^{-2}),\ R \to \infty, \tag{91}$$

(*not* $0(R^{-3})$, since the fluid is unbounded). Here $\mathbf{B}$ is the magnetic field, Φ is the electrostatic potential measured in units of ULB (U and L are a characteristic velocity and length associated with the system), and $\mathbf{u}$ is the fluid velocity measured in units of U. We assume that $\mathbf{u}$ vanishes as $R \to \infty$. The characteristic values Rm is sometimes called a "magnetic Reynolds number", and is defined by

$$\mathrm{Rm} = \mathrm{UL}/\lambda,$$

where λ is the "magnetic diffusivity" which, in electromagnetic units, is equal to $1/4\pi\sigma\mu$ (σ and μ are the conductivity and permeability of the fluid).

The adjoint problem is

$$\mathscr{L}^*:\begin{cases}\operatorname{curl} \mathbf{B}^* = \mathrm{Rm}^* \,(-\operatorname{grad} \Phi - \mathbf{u} \times \mathbf{B}^*), & (89^*)\\ \operatorname{div} \mathbf{B}^* = 0, & (90^*)\end{cases}$$

$$\mathscr{B}^*: \qquad \mathbf{B}^* = O(R^{-2}),\, R \to \infty, \qquad (91^*)$$

This can easily be proved. By the divergence theorem and equations (91), (91*), we have

$$\int \mathbf{B}^* \cdot \operatorname{curl} \mathbf{B}\, d\tau = \int \mathbf{B} \cdot \operatorname{curl} \mathbf{B}^*\, d\tau. \qquad (92)$$

But, by equations (89), and (90) and another application of the divergence theorem, the left-hand side of equation (92) is equal to

$$\mathrm{Rm} \int \mathbf{u} \cdot (\mathbf{B} \times \mathbf{B}^*)\, d\tau,$$

and, similarly, the right-hand side is equal to

$$\mathrm{Rm}^* \int \mathbf{u} \cdot (\mathbf{B} \times \mathbf{B}^*)\, d\tau$$

Thus, all the integrals concerned vanish unless Rm and Rm are equal, and then we have

$$\mathrm{Rm} = \frac{\int \mathbf{B}^* \cdot \operatorname{curl} \mathbf{B}\, d\tau}{\int \mathbf{u} \cdot (\mathbf{B} \times \mathbf{B}^*)\, d\tau}. \qquad (93)$$

The variational principle follows as before.

It is interesting to notice that, by proving that Rm and Rm* are equal, we have shown that, if dynamo action is possible in an infinite fluid with a certain velocity field $\mathbf{u}$, then it is also possible with the velocity field $-\,\mathbf{u}$. This is certainly not to be expected for a fluid of finite extent. It is not, therefore, surprising to find that, for a fluid dynamo of finite extent, equations (89*) to (91*) do not define a characteristic value problem that is adjoint to that of equations (89) to (91). In fact, there seems to be no obvious way of constructing the adjoint problem in this case because of the discontinuity in curl **B** which is generally present at the surface of the fluid.

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